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Reduction of finite-element models of planar frames using non-linear normal modes

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Abstract

This paper is an extended version of Mazzilli et al. (Mazzilli, C.E.N., Soares, M.E.S., Baracho Neto, O.G.P., 1999. Proceedings of the American Congress of Applied Mechanics, PACAM VI, vol. 8, pp. 1589–1592, Rio de Janeiro, Brazil) which presents a powerful reduction technique in non-linear dynamics based on the combination of finite element procedures with a "non-linear" Galerkin method (Zemann, J., Steindl, A., 1996. Proceedings of the 19th International Congress of Theoretical and Applied Mechanics, Kyoto, Japan) and non-linear normal modes (Shaw, S.W., Pierre, C., 1993. Journal of Sound and Vibration 164 (1), 85–124). Its implementation, in the form of a symbolic computation code, was carried out for planar framed structures under assumptions of linear elasticity and geometrical non-linearity, according to the Bernoulli–Euler rod theory (Brasil, R.M.L.R.F., Mazzilli, C.E.N., 1993. Applied Mechanics Reviews 46 (11), S110–S117). To obtain the desired drastic reduction of degrees of freedom and the corresponding set of differential equations of motion in explicit form, it is necessary to supply as input data the displacement components of the pre-selected non-linear normal modes.

Validation tests for non-linear free-vibration problems are shown, considering reduced models of higher hierarchy and their ability to supply accurate regenerated non-linear normal modes. For non-linear forced vibration problems, a brief outlook of what is intended to be done is presented.

Keywords: Non-linear dynamics; Reduction technique; Non-linear modes

1. Introduction

Important accomplishments have recently been achieved within applied mechanics with respect to the study of non-linear dynamics for few-degree-of-freedom systems. They included new geometrical, computational and analytical techniques (Thompson and Stewart, 1986; Thompson and Bishop, 1994). Computational mechanics has also developed efficient procedures for non-linear numerical analysis of large-size structural models, especially with the finite-element method (Bathe, 1982; Belytschko and Hughes, 1983).

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Yet, in the engineering practice, comprehensive surveys of non-linear responses are not usually pursued for large-size non-linear systems (Symonds and Yu, 1985; Fanelli, 1991), because most of time engineers are neither convinced of their importance nor have the means to perform them. In fact, extension of techniques from lower to higher phase-space dimensions seems to be unfeasible. What appears to be promising is the opposite, i.e., the reduction of typical large-size engineering models to a manageable small size, by taking advantage of modern non-linear modal analysis and, possibly, of computational techniques for location of invariant manifolds and their tangencies (Thompson and Bishop, 1994).

The technique proposed herewith may represent a remarkable progress towards this goal. It combines the finite-element analysis and a non-conventional mode superposition method. The starting point is an explicit non-linear formulation of dynamics, such as that presented in Brasil and Mazzilli (1993) and Mazzilli (1994) for planar frames. In most of the paper content, the approach will be illustrated for planar frames. Yet, they have a much broader application.

It will be assumed that both the deformed static equilibrium configuration and the non-linear modal displacement field, expressed in the form of power series of the modal variables, are available for the finite-element model under investigation. The (several) original generalised coordinates are then expressed as their equilibrium values plus the non-linear displacement field, which depends on (few) selected modal variables. Modal variables are still to be determined by means of time integration of the fully non-linear reduced equations of motion. The procedure developed is able to automatically render the reduced equations of motion, i.e., the reduced matrices of mass, inertial damping and stiffness, and also the equivalent applied load vector.

The elemental inertia, damping and elastic force vectors defined in the local system are then written in terms of the selected modal variables and their derivatives. Next, the elemental force vectors are transformed onto the global system, still keeping their original ranks. Reduction is then performed at element level, by pre-multiplication by the transpose of the non-linear modal matrix, as it is typical of the conventional mode superposition method. The reduced elemental force vectors are, of course, non-linearly dependent of the modal variables and their time derivatives. The overall system reduced force vectors are now easily obtained after direct summation of the corresponding elemental vectors. More concisely, the reduction can be performed at the element local systems, provided that the non-linear modes are transformed from the global onto the local systems.

The generated few-degree-of-freedom non-linear system can then be studied with the well established techniques of applied mechanics. The associated reduced system will serve to guide the relevant numerical essays, which should be finally performed for the original finite-element model.

Although the reduction technique is applicable both to free or forced-vibration problems, the numerical examples in this paper will consider only the former situation.

2. The equations of motion

The equations of motion used in this paper result from the finite-element formulation of non-linear dynamics of planar frames presented in Brasil and Mazzilli (1993), based on Bernoulli–Euler beam theory.

Fig. 1 shows how the six degrees of freedom of the element are numbered in the local coordinate system Oxy. In what follows, ℓ is the length of the element, A and I are, respectively, its cross-sectional area and moment of inertia, ρ is the mass density and E is the Young modulus. In this formulation, a concentrated mass M_0 with rotational inertia I_0 can be included at position $x = \bar{x}$.

In the local system, the equations of motion of an isolated single element are

$$\bar{m}_{rs}\ddot{\bar{q}}_s + \bar{d}_{rs}\dot{\bar{q}}_s + \bar{u}_r = \bar{f}_r, \quad r, s = 1, \ldots, 6, \tag{1}$$

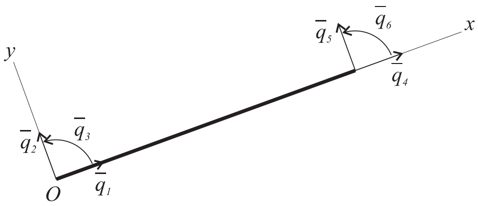

Fig. 1. Definition of degrees-of-freedom in the local system.

where $\bar{m}_{rs}$, $\bar{d}_{rs}$ and $\bar{u}_r$, $r,s = 1,\ldots,6$ represent, respectively, the elements of the mass matrix, the damping matrix and the elastic force vector components. For free vibrations, the applied nodal force vector components $\bar{f}_r$ are null.

As demonstrated in Brasil and Mazzilli (1993), the elements of the mass matrix are determined by

$$\bar{m}_{rs} = \bar{m}^0_{rs} + \left(v^i_{rs} + v^i_{sr}\right)\bar{q}_i + \left(\sigma^{ij}_{rs} + \tau^{ij}_{rs}\right)\bar{q}_i\bar{q}_j, \tag{2}$$

where

$$\bar{m}^0_{rs} = \rho A \int_0^\ell \Big[\phi_r(x)\phi_s(x) + \psi_r(x)\psi_s(x)\Big]\,\mathrm{d}x + \rho I \int_0^\ell \psi'_r(x)\psi'_s(x)\,\mathrm{d}x + M_0\Big[\phi_r(\bar{x})\phi_s(\bar{x}) + \psi_r(\bar{x})\psi_s(\bar{x})\Big] + I_0\psi'_r(\bar{x})\psi'_s(\bar{x}), \tag{3}$$

the shape functions being

$$\begin{array}{ll}
\psi_1(x) = 0, & \phi_1(x) = 1 - \frac{x}{\ell}, \\
\psi_2(x) = 1 - 3\frac{x^2}{\ell^2} + 2\frac{x^3}{\ell^3}, & \phi_2(x) = 0, \\
\psi_3(x) = x - 2\frac{x^2}{\ell} + \frac{x^3}{\ell^2}, & \phi_3(x) = 0, \\
\psi_4(x) = 0, & \phi_4(x) = \frac{x}{\ell}, \\
\psi_5(x) = 3\frac{x^2}{\ell^2} - 2\frac{x^3}{\ell^3}, & \phi_5(x) = 0, \\
\psi_6(x) = -\frac{x^2}{\ell} + \frac{x^3}{\ell^2}, & \phi_6(x) = 0.
\end{array} \tag{4}$$

The other coefficients in Eq. (2) are calculated using

$$v^i_{rs} = \rho A \int_0^\ell \phi_r(x)\beta_{is}(x)\,\mathrm{d}x + M_0\phi_r(\bar{x})\beta_{is}(\bar{x}), \tag{5}$$

$$\sigma^{ij}_{rs} = \rho A \int_0^\ell \beta_{ir}(x)\beta_{js}(x)\,\mathrm{d}x + M_0\beta_{ir}(\bar{x})\beta_{js}(\bar{x}), \tag{6}$$

$$\tau^{ij}_{rs} = \rho I \int_0^\ell \psi'_r(x)\psi'_s(x)\psi'_i(x)\psi'_j(x)\,\mathrm{d}x, \tag{7}$$

where

$$\beta_{ij}(x) = \frac{x}{\ell}\alpha_{ij}(\ell) - \alpha_{ij}(x), \tag{8}$$

with

$$\alpha_{ij}(x) = \int_0^x \psi'_i(z)\psi'_j(z)\,\mathrm{d}z. \tag{9}$$

Note that, in Eq. (2), $\bar{m}^0_{rs}$, $r,s = 1,\dots,6$ are the elements of the constant mass matrix used in linear dynamics, including the effect of concentrated mass and rotational inertia. Besides the usual longitudinal displacement interpolation of linear theory, geometrical non-linearities are introduced under the hypothesis of invariance of axial force inside the finite element.

The damping matrix elements are determined by

$$\bar{d}_{rs} = \mu_{rs} + v^i_{rs}\dot{\bar{q}}_i + \left(\sigma^{ji}_{rs} + \tau^{ij}_{rs}\right)\dot{\bar{q}}_i\bar{q}_j, \tag{10}$$

where μ_{rs}, $r,s = 1,\dots,6$ are the damping matrix elements used in linear dynamics, whose expression depends on the kind of damping considered; the remaining terms have inertial origin.

Finally, the elastic force vector can be written as

$$\bar{u}_r = \bar{u}^0_{rs}\bar{q}_s + \left(\tfrac{1}{2}\rho^i_{rs} + \rho^s_{ir}\right)\bar{q}_i\bar{q}_s + \tfrac{1}{2}\theta^{ij}_{rs}\bar{q}_i\bar{q}_j\bar{q}_s, \tag{11}$$

where

$$\bar{u}^0_{rs} = EA\ell\phi'_r\phi'_s + EI\int_0^\ell \psi''_r(x)\psi''_s(x)\,\mathrm{d}x, \tag{12}$$

$$\rho^i_{rs} = EA\phi'_r\alpha_{is}(\ell), \tag{13}$$

$$\theta^{ij}_{rs} = \frac{EA}{\ell}\alpha_{ij}(\ell)\alpha_{rs}(\ell). \tag{14}$$

Eq. (12) shows that $\bar{u}^0_{rs}$, $r,s = 1,\dots,6$ are the elements of the classical stiffness matrix.

After assemblage of the equations corresponding to individual elements, the equations of motion for the entire structure subjected to dynamic excitation can be written as

$$M_{ij}(\mathbf{p})\ddot{p}_j + D_{ij}(\mathbf{p},\dot{\mathbf{p}})\dot{p}_j + U_i(\mathbf{p}) = F_{0i} + F_i(t), \quad i,j = 1,\dots,n, \tag{15}$$

where $\mathbf{F}_0$ and $\mathbf{F}(t)$ stand for the static and dynamic loading, respectively.

The equilibrium configuration $\mathbf{p}_0$ is the solution of

$$U_i(\mathbf{p}_0) = K_{ij}(\mathbf{p}_0)p_{0j} = F_{0i}, \quad i,j = 1,\dots,n, \tag{16}$$

where $K_{ij}(\mathbf{p}_0)$ is an element of the secant stiffness matrix.

Let $\mathbf{p}^*$ be

$$\mathbf{p}^* = \mathbf{p} - \mathbf{p}_0. \tag{17}$$

The equations of motion can be re-written for $\mathbf{p}^*$ as

$$\left[M_{ij}(\mathbf{p}_0 + \mathbf{p}^*)\right]\ddot{p}^*_j + \left[D_{ij}(\mathbf{p}_0 + \mathbf{p}^*, \dot{\mathbf{p}}^*)\right]\dot{p}^*_j + \left[K_{ij}(\mathbf{p}_0 + \mathbf{p}^*)\right](p_{0j} + p^*_j) = F_{0i} + F_i(t), \quad i,j = 1,\dots,n. \tag{18}$$

For free vibrations about the equilibrium configuration, it follows that

$$M^*_{ij}(\mathbf{p}_0, \mathbf{p}^*)\ddot{p}^*_j + D^*_{ij}(\mathbf{p}_0, \mathbf{p}^*, \dot{\mathbf{p}}^*)\dot{p}^*_j + K^*_{ij}(\mathbf{p}_0, \mathbf{p}^*)p^*_j = 0, \quad i,j = 1,\dots,n. \tag{19}$$

When vibrations are considered about the undeformed configuration, Eqs. (18) and (19) simplify to

$$M^*_{ij}(\mathbf{p}^*)\ddot{p}^*_j + D^*_{ij}(\mathbf{p}^*, \dot{\mathbf{p}}^*)\dot{p}^*_j + K^*_{ij}(\mathbf{p}^*)p^*_j = 0, \quad i,j = 1,\dots,n, \tag{20}$$

where

$$\begin{aligned} M^*_{ij}(\mathbf{p}^*) &= M^0_{ij} + M^1_{ijk}p^*_k + M^2_{ijkl}p^*_k p^*_l, \\ D^*_{ij}(\mathbf{p}^*, \dot{\mathbf{p}}^*) &= D^0_{ij} + D^1_{ijk}\dot{p}^*_k + D^2_{ijkl}\dot{p}^*_k p^*_l, \\ K^*_{ij}(\mathbf{p}^*) &= K^0_{ij} + K^1_{ijk}p^*_k + K^2_{ijkl}p^*_k p^*_l \end{aligned} \tag{21}$$

and M^0_{ij}, M^1_{ijk}, M^2_{ijkl}, D^0_{ij}, D^1_{ijk}, D^2_{ijkl}, K^0_{ij}, K^1_{ijk}, K^2_{ijkl}, $i, j, k, l = 1, \ldots, n$ are constants.

3. Non-linear normal modes

Non-linear normal modes were originally defined as synchronous motions exhibited by non-linear autonomous systems in which there are fixed, usually non-linear, relations between generalised coordinates Rosenberg (1966). In that sense, the classical normal modes of linear systems, for which these relations are linear, represent a particular case of a more general concept.

Shaw and Pierre (1993) proposed a new definition based on geometrical properties of a modal solution trajectory in the system phase space. By noticing that, during a modal motion of a linear system with n degrees-of-freedom, this trajectory is confined to a two-dimensional plane surface in R^{2n}, they redefined a normal mode as a *motion which takes place on a two-dimensional invariant manifold in the system phase space*. This definition applies to weakly non-linear oscillatory systems as well, and is equally suited to non-conservative problems.

During such a motion, every generalised displacement or velocity can be written as a function of two of them, under certain non-degeneracy conditions. In the non-linear case, the invariant manifolds may be slightly curved; as a consequence, the functions relating generalised displacements and velocities during modal motions may be non-linear.

Consider a non-linear system with n degrees-of-freedom $p_1, \ldots, p_n$ governed by the first-order equations of motion:

$$\begin{aligned} \dot{p}_i &= w_i, \\ \dot{w}_i &= g_i(p_1, \ldots, p_n, w_1, \ldots, w_n), \quad i = 1, \ldots, n, \end{aligned} \tag{22}$$

where g_i, $i = 1, \ldots, n$ are analytical functions such that

$$g_i(0, \ldots, 0, 0, \ldots, 0) = 0, \quad i = 1, \ldots, n. \tag{23}$$

Suppose that, when linearised about the equilibrium position $(\mathbf{p}, \mathbf{w}) = \mathbf{0}$, the system has distinct pairs of complex conjugate eigenvalues and eigenvectors, typical of an oscillatory behaviour.

We expect to find n two-dimensional invariant manifolds in the system phase space, each of them associated with a particular normal mode and, consequently, with a set of functions relating all generalised coordinates and velocities to two of them. If we choose p_k and w_k as independent variables and denote them by Y and Z, respectively, the modal relations we are looking for may be expressed as

$$\begin{aligned} p_i(t) &= P_i(Y(t), Z(t)), \\ w_i(t) &= W_i(Y(t), Z(t)), \quad i = 1, \ldots, n, \end{aligned} \tag{24}$$

where P_i, W_i, $i = 1, \ldots, n$ are supposed to be analytical functions. It is easy to see that, in particular,

$$P_k(Y, Z) = Y, \qquad W_k(Y, Z) = Z. \tag{25}$$

The substitution of Eq. (24) into Eq. (22) leads to

$$\begin{aligned}
&\frac{\partial P_i}{\partial Y} Z + \frac{\partial P_i}{\partial Z} g_k(P_1,\ldots,P_n,W_1,\ldots,W_n) = W_i, \\
&\frac{\partial W_i}{\partial Y} Z + \frac{\partial W_i}{\partial Z} g_k(P_1,\ldots,P_n,W_1,\ldots,W_n) = g_i(P_1,\ldots,P_n,W_1,\ldots,W_n), \quad i = 1,\ldots,n,
\end{aligned} \tag{26}$$

which is a non-linear system of partial differential equations having, as unknowns, the modal relations. Each solution to this system geometrically describes one of the invariant manifolds.

In most cases, it is impossible to find out the exact solutions of Eq. (26), and a power series approximation is needed. Consider the first-order equations of motion written as cubic expansions, by means of

$$\begin{aligned}
g_i(p_1,\ldots,p_n,w_1,\ldots,w_n) = {}& B_{ij}p_j + C_{ij}w_j + E_{ijm}p_jp_m + F_{ijm}p_jw_m + G_{ijm}w_jw_m + H_{ijmp}p_jp_mp_p \\
&+ L_{ijmp}p_jp_mw_p + N_{ijmp}p_jw_mw_p + R_{ijmp}w_jw_mw_p,
\end{aligned} \tag{27}$$

where B_{ij}, C_{ij}, E_{ijm}, F_{ijm}, G_{ijm}, H_{ijmp}, L_{ijmp}, N_{ijmp} and R_{ijmp} are known coefficients and $i,j,m,p = 1,\ldots,n$. The approximate modal relations are also written in polynomial form:

$$\begin{aligned}
P_i(Y,Z) &= a_{1i}Y + a_{2i}Z + a_{3i}Y^2 + a_{4i}YZ + a_{5i}Z^2 + a_{6i}Y^3 + a_{7i}Y^2Z + a_{8i}YZ^2 + a_{9i}Z^3, \\
W_i(Y,Z) &= b_{1i}Y + b_{2i}Z + b_{3i}Y^2 + b_{4i}YZ + b_{5i}Z^2 + b_{6i}Y^3 + b_{7i}Y^2Z + b_{8i}YZ^2 + b_{9i}Z^3, \quad i = 1,\ldots,n,
\end{aligned} \tag{28}$$

where a_{ji}, b_{ji}, $j = 1,\ldots,9$, $i = 1,\ldots,n$ are constants to be determined.

After substituting Eqs. (28) and (27) in Eq. (26), and collecting terms of equal order in Y and Z in the resulting polynomial equations, a large system of non-linear algebraic equations having the coefficients a_{ji}, b_{ji}, $j = 1,\ldots,9$, $i = 1,\ldots,n$ as unknowns is constructed. There must be n different solutions to this system, corresponding to the n distinct invariant manifolds. It is possible to show (Shaw and Pierre, 1993) that these equations can be ordered in such a manner that, instead of solving them all at once, we can solve a much smaller system of non-linear algebraic equations having as unknowns just the coefficients of the linear terms in the modal relations; after that, two *linear* systems are constructed and solved, one for the coefficients of quadratic terms and the other for the coefficients of cubic terms.

In fact, the solution to the first (non-linear) system of algebraic equations can be avoided (Soares and Mazzilli, 1999). The unknown coefficients a_{1i}, a_{2i}, b_{1i}, b_{2i}, $i = 1,\ldots,n$ describe a two-dimensional planar surface in the phase space which is tangent to the corresponding curved invariant manifold at the equilibrium point; this planar surface coincides with the invariant manifold of the linearised system, thus related to the eigenvectors of the mode of interest. Hence, the linear part of the modal relations can be alternatively generated from the solution to an eigenvalue problem.

In this paper, the invariant manifold approach was adopted to generate the non-linear modes used in the proposed reduction technique. The pioneering implementation described in Soares and Mazzilli (1999) is capable of generating non-linear normal modes of finite-element models. It was applied to systems (20) and (21), which assumes free vibrations about the undeformed configuration. This assumption is acceptable for a great number of cases, such as those of Section 6.

The system of n second-order equations had to be transformed into a system of $2n$ first-order equations like Eq. (22). In the theory, this can be accomplished by simply solving Eq. (20) in terms of the accelerations and expanding the result in power series. However, the operation involves a symbolic inversion of the non-constant mass matrix, and this is not feasible in practice.

Again, we can find an approximate solution to this problem by using the Taylor series. Substituting the desired expanded form (27) of the equations of motion (22) into the second-order system (20) and equating coefficients of like powers of p_i and $w_i = \dot{p}_i$, we arrive at a system of linear algebraic equations having as unknowns the coefficients B_{ij}, C_{ij}, E_{ijm}, F_{ijm}, G_{ijm}, H_{ijmp}, L_{ijmp}, N_{ijmp} and R_{ijmp} $(i,j,m,p = 1,\ldots,n)$.

Once known a particular set of modal relations (28), the dynamics on the corresponding invariant manifold can be generated by substituting them in the kth pair of equations of motion (22), considering expansion (27), and solving the resulting modal oscillator, generally non-linear, to obtain $Y(t)$ and $Z(t)$.

4. Non-linear modal displacements

The equations of motion for a scleronomic Lagrangian system such as that discussed in Section 2 may be written, in matrix form, as

$$\mathbf{M}(\mathbf{p})\ddot{\mathbf{p}} + \mathbf{D}(\mathbf{p}, \dot{\mathbf{p}})\dot{\mathbf{p}} + \mathbf{U}(\mathbf{p}) = \mathbf{F}, \tag{29}$$

where $\mathbf{M}$ is the secant matrix of mass, $\mathbf{D}$, the secant matrix of equivalent damping, $\mathbf{U}$, the elastic force vector, $\mathbf{F}$, the generalised applied force vector and $\mathbf{p}$, the generalised coordinate (displacement) vector.

Let $\mathbf{p}_0$ be the displacement vector for the deformed equilibrium configuration, determined from a non-linear static analysis, if necessary. The *rth component* of the displacement vector is given by

$$p_r = p_{0r} + \xi_r^u(\mathbf{p}_0, Y_1, Y_2, \ldots)Y_u, \quad \text{sum in } u = 1, 2, 3, \ldots, \tag{30}$$

where Y_u stands for the *uth modal variable*, which is a new generalised coordinate, and $\xi_r^u(\mathbf{p}_0, Y_1, Y_2, \ldots)$ *the non-linear function of the modal variables associated with the rth displacement component* and the *uth mode*. Note that ξ_r^u represents the displacement content of the non-linear mode u. It is assumed that the velocity content of the non-linear mode u is of little relevance and therefore, it has not been taken into account in the current reduction technique version. Section 6 is concerned with the validation of the procedure as a whole, including this hypothesis. It is further supposed that these functions ξ_r^u can be defined by power series of the modal variables:

$$\xi_r^u(\mathbf{p}_0, Y_1, Y_2, \ldots) = \xi_r^{0u}(\mathbf{p}_0) + \xi_r^{uv}(\mathbf{p}_0)Y_v + \xi_r^{uvw}(\mathbf{p}_0)Y_vY_w + \cdots \tag{31}$$

Taking Eq. (30) into account, it is possible to write for the displacement increment vector component,

$$\delta p_r = \delta\xi_r^u Y_u + \xi_r^u \delta Y_u. \tag{32}$$

As a consequence of Eq. (31), one may re-write

$$\delta p_r = \Phi_r^u \delta Y_u, \tag{33}$$

where

$$\Phi_r^u = \Phi_r^{0u} + \Phi_r^{uv}Y_v + \Phi_r^{uvw}Y_vY_w + \cdots \tag{34}$$

and

$$\begin{aligned} \Phi_r^{0u} &= \xi_r^{0u}, \\ \Phi_r^{uv} &= \xi_r^{uv} + \xi_r^{vu}, \\ \Phi_r^{uvw} &= \xi_r^{uvw} + \xi_r^{vuw} + \xi_r^{wvu}. \end{aligned} \tag{35}$$

The functions Φ_r^u will be termed here as the *non-linear function of the modal variables associated with the rth displacement-increment component of the uth mode*. It is assumed that such non-linear functions are known. Once each one of Φ_r^u is known, ξ_r^u is also known using Eq. (35) and vice versa.

It should be observed that the non-linear modal displacement fields, introduced as above, must be kinematically admissible and have embedded in themselves, the usual linear modes about the deformed configuration, determined in a classical eigenvalue problem.

It will not be necessarily required here the uncoupling of non-linear terms, as it is implied in the definition of non-linear normal modes (Shaw and Pierre, 1993). In fact, if non-linear multimodes should be considered, e.g. when internal resonance conditions apply and non-linear normal modes cannot be determined, the reduction technique will still be able to handle the associated modal variable couplings, provided such multimodes are supplied.

5. The reduction technique

The procedure allows one to automatically obtain the reduced equations of motion for the structural system:

$$\mathbf{M}^{**}(\mathbf{Y})\ddot{\mathbf{Y}} + \mathbf{D}^{**}(\mathbf{Y},\dot{\mathbf{Y}})\dot{\mathbf{Y}} + \mathbf{U}^{**}(\mathbf{Y},\dot{\mathbf{Y}},\ddot{\mathbf{Y}}) = \mathbf{F}^{**}, \tag{36}$$

where $\mathbf{M}^{**}$ and $\mathbf{D}^{**}$ are, respectively, the reduced secant matrices of mass and equivalent damping, and $\mathbf{U}^{**}$ and $\mathbf{F}^{**}$ are, respectively, the reduced elastic and applied load vectors.

At the finite-element level, the corresponding matrices and vectors are explicitly known. In particular, Section 2 supplies the mass matrix $\bar{\mathbf{m}}$, the equivalent secant damping matrix $\bar{\mathbf{d}}$ and the elastic force vector $\bar{\mathbf{u}}$ in the local system for the Bernoulli–Euler rod finite element. They are defined by non-linear functions of the element local system displacement and velocity vectors ($\bar{\mathbf{q}}$ and $\dot{\bar{\mathbf{q}}}$). It should also be noted that the $\bar{\mathbf{u}}$ vector components are evaluated as the partial derivatives of the strain energy with respect to the components of the displacement vector $\bar{\mathbf{q}}$.

Elemental matrices and vectors in the local system will then be re-written in terms of the selected modal variables and their time derivatives, making use of the rotation matrix $\mathbf{T}$ and the matrix ζ, which is the partition of ξ associated with the particular finite element under consideration. In fact, it is known that the element displacement vector in the global system is given by

$$\mathbf{q} = \mathbf{q}_0 + \zeta\mathbf{Y}, \tag{37}$$

where $\mathbf{q}_0$ stands for the equilibrium configuration displacement vector. In the local system, the corresponding equation will read

$$\bar{\mathbf{q}} = \bar{\mathbf{q}}_0 + \bar{\zeta}\mathbf{Y}, \tag{38}$$

where

$$\begin{aligned}\bar{\mathbf{q}}_0 &= \mathbf{T}\mathbf{q}_0,\\ \bar{\zeta} &= \mathbf{T}\zeta.\end{aligned} \tag{39}$$

It is therefore possible to express the elemental matrices and vectors in the local system in terms of the modal variables, provided an explicit formulation such as that of Brasil and Mazzilli (1993) is at reach. We thus write for each element

$$\begin{aligned}\bar{\mathbf{m}}(\bar{\mathbf{q}}) &= \bar{\mathbf{m}}(\bar{\mathbf{q}}_0 + \bar{\zeta}\mathbf{Y}) = \bar{\mathbf{m}}^*(\mathbf{Y}),\\ \bar{\mathbf{d}}(\bar{\mathbf{q}},\dot{\bar{\mathbf{q}}}) &= \bar{\mathbf{d}}(\bar{\mathbf{q}}_0 + \bar{\zeta}\mathbf{Y}, \dot{\bar{\zeta}}\mathbf{Y} + \bar{\zeta}\dot{\mathbf{Y}}) = \bar{\mathbf{d}}^*(\mathbf{Y},\dot{\mathbf{Y}}),\\ \bar{\mathbf{u}}(\bar{\mathbf{q}}) &= \bar{\mathbf{u}}(\bar{\mathbf{q}}_0 + \bar{\zeta}\mathbf{Y}) = \bar{\mathbf{u}}^*(\mathbf{Y}),\end{aligned} \tag{40}$$

where the time derivatives of the ζ functions, up to quadratic terms, are

$$\begin{aligned}\dot{\bar{\zeta}}_r^u &= \bar{\zeta}_r^{uv}\dot{Y}_v + \left(\bar{\zeta}_r^{uvw} + \bar{\zeta}_r^{uwv}\right)\dot{Y}_v Y_w,\\ \ddot{\bar{\zeta}}_r^u &= \bar{\zeta}_r^{uv}\ddot{Y}_v + \left(\bar{\zeta}_r^{uvw} + \bar{\zeta}_r^{uwv}\right)\ddot{Y}_v Y_w + \left(\bar{\zeta}_r^{uvw} + \bar{\zeta}_r^{uwv}\right)\dot{Y}_v\dot{Y}_w.\end{aligned} \tag{41}$$

Elemental force vectors are then evaluated in terms of the selected modal variables:

$$\begin{aligned}\bar{\mathbf{m}}(\bar{\mathbf{q}})\ddot{\bar{\mathbf{q}}} &= \bar{\mathbf{m}}^*(\mathbf{Y})(\ddot{\bar{\zeta}}\mathbf{Y} + 2\dot{\bar{\zeta}}\dot{\mathbf{Y}} + \bar{\zeta}\ddot{\mathbf{Y}}),\\ \bar{\mathbf{d}}(\bar{\mathbf{q}},\dot{\bar{\mathbf{q}}})\dot{\bar{\mathbf{q}}} &= \bar{\mathbf{d}}^*(\mathbf{Y},\dot{\mathbf{Y}})(\dot{\bar{\zeta}}\mathbf{Y} + \bar{\zeta}\dot{\mathbf{Y}}).\end{aligned} \tag{42}$$

Next, elemental force vectors are transformed onto the global system, by pre-multiplication by the transpose of the rotation matrix **T**, still keeping their original rank:

$$\begin{aligned} \mathbf{T}^t\bar{\mathbf{m}}(\bar{\mathbf{q}})\ddot{\bar{\mathbf{q}}} &= \mathbf{T}^t\bar{\mathbf{m}}^*(\mathbf{Y})(\ddot{\bar{\zeta}}\mathbf{Y} + 2\dot{\bar{\zeta}}\dot{\mathbf{Y}} + \bar{\zeta}\ddot{\mathbf{Y}}),\\ \mathbf{T}^t\bar{\mathbf{d}}(\bar{\mathbf{q}},\dot{\bar{\mathbf{q}}})\dot{\bar{\mathbf{q}}} &= \mathbf{T}^t\bar{\mathbf{d}}^*(\mathbf{Y},\dot{\mathbf{Y}})(\dot{\bar{\zeta}}\mathbf{Y} + \bar{\zeta}\dot{\mathbf{Y}}),\\ \mathbf{T}^t\bar{\mathbf{u}}(\bar{\mathbf{q}}) &= \mathbf{T}^t\bar{\mathbf{u}}^*(\mathbf{Y}),\\ \mathbf{f}^* &= \mathbf{T}^t\bar{\mathbf{f}}^*, \end{aligned} \tag{43}$$

where $\mathbf{f}^*$ and $\bar{\mathbf{f}}^*$ stand for the elemental applied force vectors in the global and local systems, respectively.

Reduction is then executed for each element, by pre-multiplication of each vector in the global system by the transpose of the "modal" matrix, as it is typical of the conventional mode superposition method. In fact, let $\mathbf{F_q}$ be any force of the original global equation of motion and $\mathbf{F_Y}$ be the associated force of the reduced equation of motion. The virtual work $\delta\mathscr{W}$ produced should be equal for both forces. Hence,

$$\delta\mathscr{W} = \delta\mathbf{q}^t\mathbf{F_q} = \delta\mathbf{Y}^t\mathbf{F_Y}. \tag{44}$$

By taking into account that

$$\delta\mathbf{q} = \phi\,\delta\mathbf{Y}, \tag{45}$$

where ϕ is the partition of Φ associated with the particular finite element under consideration, it is straightforward that the reduced force must be given by

$$\mathbf{F_Y} = \phi^t\mathbf{F_q}. \tag{46}$$

One may thus re-arrange the elemental reduced equations in the form

$$\mathbf{m}^{**}(\mathbf{Y})\ddot{\mathbf{Y}} + \mathbf{d}^{**}(\mathbf{Y},\dot{\mathbf{Y}})\dot{\mathbf{Y}} + \mathbf{u}^{**}(\mathbf{Y},\dot{\mathbf{Y}},\ddot{\mathbf{Y}}) = \mathbf{f}^{**}, \tag{47}$$

where

$$\begin{aligned} \mathbf{m}^{**}(\mathbf{Y}) &= \phi^t\mathbf{T}^t\bar{\mathbf{m}}^*(\mathbf{Y})\bar{\zeta},\\ \mathbf{d}^{**}(\mathbf{Y},\dot{\mathbf{Y}}) &= \phi^t\mathbf{T}^t[\bar{\mathbf{d}}^*(\mathbf{Y},\dot{\mathbf{Y}})\bar{\zeta} + 2\bar{\mathbf{m}}^*(\mathbf{Y})\dot{\bar{\zeta}}],\\ \mathbf{u}^{**}(\mathbf{Y},\dot{\mathbf{Y}},\ddot{\mathbf{Y}}) &= \phi^t\mathbf{T}^t\{\bar{\mathbf{u}}^* + [\bar{\mathbf{m}}^*(\mathbf{Y})\ddot{\bar{\zeta}} + \bar{\mathbf{d}}^*(\mathbf{Y},\dot{\mathbf{Y}})\dot{\bar{\zeta}}]\mathbf{Y}\},\\ \mathbf{f}^{**}(\mathbf{Y}) &= \phi^t\mathbf{T}^t\bar{\mathbf{f}}^*. \end{aligned} \tag{48}$$

The resulting matrices and vectors are then added for all elements to render the overall structure reduced matrices and vectors $\mathbf{M}^{**}$, $\mathbf{D}^{**}$, $\mathbf{U}^{**}$ and $\mathbf{F}^{**}$, as desired:

$$\begin{aligned} \mathbf{M}^{**}(\mathbf{Y}) &= \sum_e \mathbf{m}^{**}(\mathbf{Y}),\\ \mathbf{D}^{**}(\mathbf{Y},\dot{\mathbf{Y}}) &= \sum_e \mathbf{d}^{**}(\mathbf{Y},\dot{\mathbf{Y}}),\\ \mathbf{U}^{**}(\mathbf{Y},\dot{\mathbf{Y}},\ddot{\mathbf{Y}}) &= \sum_e \mathbf{u}^{**}(\mathbf{Y},\dot{\mathbf{Y}},\ddot{\mathbf{Y}}),\\ \mathbf{F}^{**}(\mathbf{Y}) &= \sum_e \mathbf{f}^{**}(\mathbf{Y}). \end{aligned} \tag{49}$$

Alternatively, the reduction can be performed directly in the element local systems, provided that the non-linear ϕ functions are expressed in the local systems themselves, i.e.

$$\bar{\phi} = \mathbf{T}\phi. \tag{50}$$

In such a case, one may also evaluate $\mathbf{m}^{**}$, $\mathbf{d}^{**}$, $\mathbf{u}^{**}$ and $\mathbf{f}^{**}$ according to

$$
\begin{aligned}
\mathbf{m}^{**}(\mathbf{Y}) &= \bar{\phi}^t \bar{\mathbf{m}}^*(\mathbf{Y})\bar{\zeta},\\
\mathbf{d}^{**}(\mathbf{Y},\dot{\mathbf{Y}}) &= \bar{\phi}^t[\bar{\mathbf{d}}^*(\mathbf{Y},\dot{\mathbf{Y}})\bar{\zeta} + 2\bar{\mathbf{m}}^*(\mathbf{Y})\dot{\bar{\zeta}}],\\
\mathbf{u}^{**}(\mathbf{Y},\dot{\mathbf{Y}},\ddot{\mathbf{Y}}) &= \bar{\phi}^t\{\bar{\mathbf{u}}^* + [\bar{\mathbf{m}}^*(\mathbf{Y})\ddot{\bar{\zeta}} + \bar{\mathbf{d}}^*(\mathbf{Y},\dot{\mathbf{Y}})\dot{\bar{\zeta}}]\mathbf{Y}\},\\
\mathbf{f}^{**}(\mathbf{Y}) &= \bar{\phi}^t\bar{\mathbf{f}}^*.
\end{aligned}
\tag{51}
$$

The reduced equation of motion for planar frames about the equilibrium configuration for mode y can be shown to be of the form

$$
\begin{aligned}
&A_y^{\ddot{u}}\ddot{Y}_u + A_y^{\ddot{u}v}\ddot{Y}_uY_v + A_y^{\ddot{u}vw}\ddot{Y}_uY_vY_w + B_y^{\dot{u}}\dot{Y}_u + B_y^{\dot{u}\dot{v}}\dot{Y}_u\dot{Y}_v + B_y^{\dot{u}v}\dot{Y}_uY_v + B_y^{\dot{u}vw}\dot{Y}_uY_vY_w + B_y^{\dot{u}\dot{v}w}\dot{Y}_u\dot{Y}_vY_w + C_y^uY_u\\
&\quad + C_y^{uv}Y_uY_v + C_y^{uvw}Y_uY_vY_w = \Delta D_y + D_y^uY_u + D_y^{uv}Y_uY_v,
\end{aligned}
\tag{52}
$$

where the coefficients are evaluated after summation over all finite elements. Their explicit expressions in terms of the $\bar{\mathbf{q}}_0$ and $\bar{\zeta}$ components and the finite-element coefficients $\bar{m}_{rs}^0$, v_{rs}^i, σ_{rs}^{ij}, τ_{rs}^{ij}, ρ_{rs}^i and θ_{rs}^{ij} defined in Section 2 are too lengthy to show here.

6. Validation tests in non-linear free vibration

Although the technique is able to supply a reduced model under generic dynamical loading, this section will be devoted to problems of free vibration. Even in free vibration, there are validation tests which may be proposed to allow for the quality assessment of the reduced model. In fact, one might think of re-generating the non-linear modes, once the reduced model is available, for later comparison with the very non-linear modes which were used to define the reduced model itself. In other words, the previously discussed procedure for non-linear mode determination can be applied to the set of differential equations describing the reduced model and the results compared with the original modes. If the relevant non-linear modes have actually been kept in the reduced model, one could expect a good correlation. Sometimes, the equations which stand for the dynamics of the modal oscillators in their respective invariant manifolds show some differences, when we compare them term by term. Yet, if their responses (e.g. the frequency–amplitude relationships) agree well, they may be acceptable. In other cases, major discrepancies indicate the inadequacy of the reduced model, inasmuch as other modes still not considered should be taken into account. It is then necessary to re-define the reduced model, using more modes, and re-assess the regenerated non-linear modes until good correlation is achieved.

Two examples are considered next, for which conclusions regarding the quality of the reduced models are drawn. To give an idea of the processing time for a reduction of a 39 degree-of-freedom model to one of three degrees-of-freedom, roughly 2 h is required in a well configured PC, running a symbolic computation code. For the sake of simplicity, in these examples, the equilibrium configuration was considered to be approximately coincident with the undeformed configuration (i.e. $\mathbf{p}_0 = \mathbf{0}$).

6.1. Portal frame

The portal frame of Fig. 2 has been studied as a finite-element model of 14 members and 39 active degrees of freedom. Its first five non-linear modes have been determined, following the procedure described before. Non-linear normal modes were evaluated using the program *modonl* (Soares and Mazzilli, 1999). To perform model reduction, only the first two modes (Figs. 3 and 4) were initially considered. It comes out from the reduction technique that the resulting two non-linear differential equations are coupled, in spite of the original non-linear modes being orthogonal. When the regenerated modes are determined, the modal-oscillator equations obey the following general pattern:

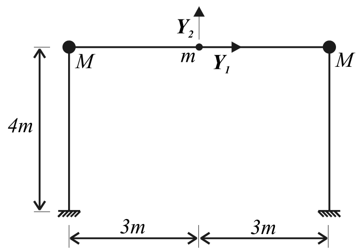

Fig. 2. Portal frame characteristics: $E = 2.0317 \times 10^{11}$ N/m^2, $\rho = 7800$ kg/m^3, $A = 2.64 \times 10^{-3}$ m^2, $I = 4.45 \times 10^{-6}$ m^4, $m = 186$ kg and $M = 140$ kg.

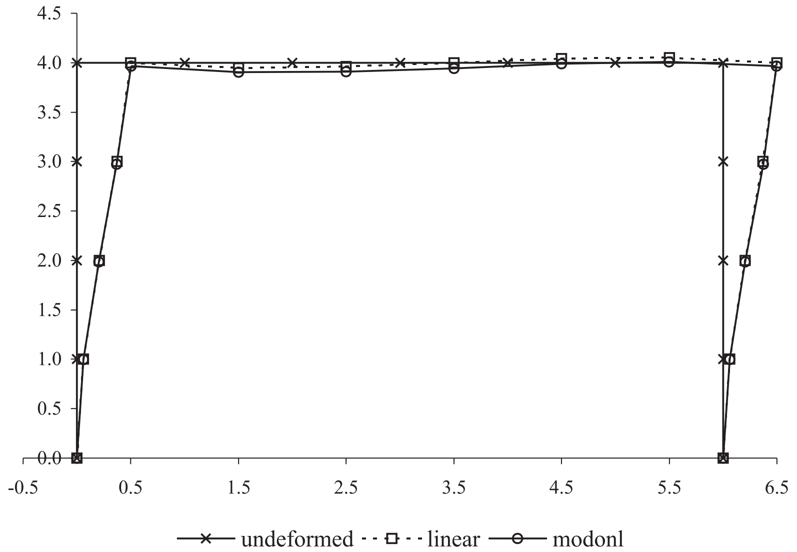

Fig. 3. First mode of the portal frame.

$$\ddot{Y} + aY + bYY + e\dot{Y}\dot{Y} + cYYY + d\dot{Y}\dot{Y}Y = 0. \tag{53}$$

The frequency–amplitude relationship can be established using the multiple scales method and it is approximately given by

$$\frac{\omega}{\sqrt{a}} = 1 - \frac{\alpha}{8a}A^2, \tag{54}$$

where α can be evaluated from the coefficients of the modal oscillator equation:

$$\alpha = \frac{4}{3a}\left[b^2 + bca + (ea)^2\right] - 3c. \tag{55}$$

The third non-linear mode (Fig. 5) was then taken into account to perform model reduction. The new regenerated non-linear modes were determined from the corresponding set of three differential equations. In what follows, particular interest is placed on the effect this third mode has on the first two regenerated modes. Tables 1 and 2 allow direct comparison among the original and regenerated modes (for both reduced models).

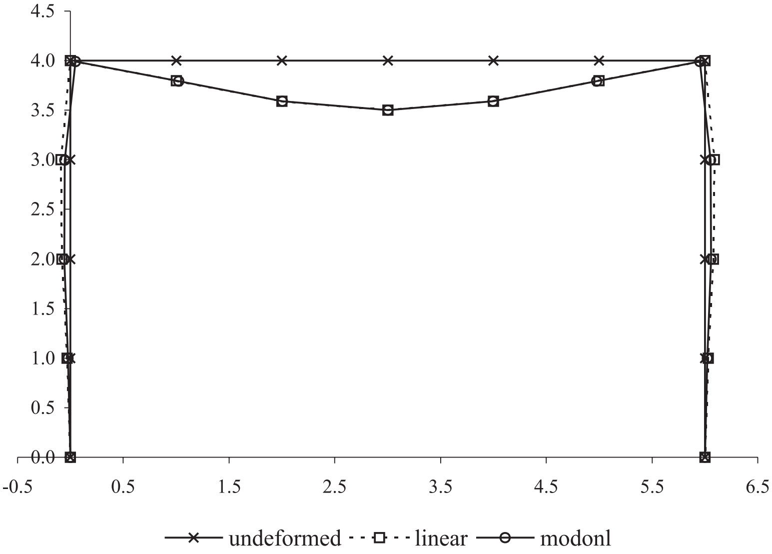

Fig. 4. Second mode of the portal frame.

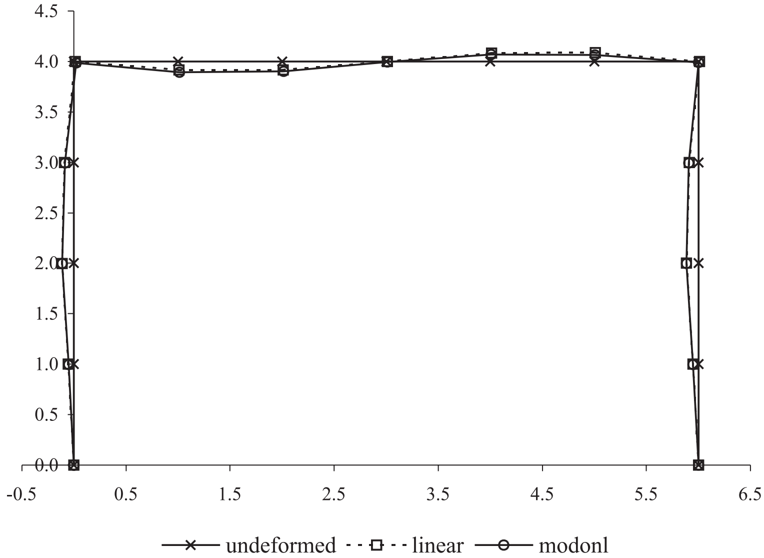

Fig. 5. Third mode of the portal frame.

The first regenerated mode, when only the first two modes were considered in the reduction technique, does not agree well with its original description. Not only are the coefficients of the modal oscillator equation in major disagreement (even with sign change for coefficient c), but also the frequency–amplitude relationship displays a large deviation (Δ). For the same reduced model, the description of the second regenerated mode is surprisingly good. When the third non-linear mode is introduced in the analysis, a remarkable improvement is seen in the first regenerated mode, although the second regenerated mode does

Table 1
Comparison among models for the first mode of the portal frame

	Original model	Regenerated (2 dof)	Regenerated (3 dof)
a	4.42×10^2	4.41×10^2	4.40×10^2
b	-5.14×10^{-4}		
e	-5.71×10^{-9}		
c	-4.64×10	8.86×10^3	-4.52×10
α	139.101	−26 573.4	135.618
$\alpha/8a$	0.0394	−7.5253	0.0385
Δ (%)		−19 200	2

Table 2
Comparison among models for the second mode of the portal frame

	Original model	Regenerated (2 dof)	Regenerated (3 dof)
a	3.41×10^3	3.42×10^3	3.41×10^3
b	9.72×10^2	9.84×10^2	9.86×10^2
e	1.24×10^{-2}	7.40×10^{-3}	1.03×10^{-2}
c	-9.85×10^2	-9.79×10^3	-1.38×10^3
α	3342.16	3322.85	4534.47
$\alpha/8a$	0.1226	0.1215	0.1663
Δ (%)		1	36

experience some loss of quality. As a whole, even with this moderate worsening in the second mode, the new description of the reduced model is considerably better than the previous one. This is especially so because, from the qualitative viewpoint, the softening effect of the original modal oscillator equations is correctly captured in the model of higher hierarchy. In addition, the Δ deviation for the second regenerated mode affects a term which represents a small correction to the linear estimate of the frequency. It is expected that the addition of the fourth mode would improve the second regenerated mode, without severely spoiling the first one.

6.2. Clamped–clamped beam

The clamped–clamped beam of Fig. 6 has been studied as a finite-element model of 20 members and 56 active degrees of freedom. Only half beam has been considered due to model symmetry. Its first five symmetric non-linear modes have been determined, following the procedure described before. In Fig. 7, the first non-linear mode, as computed by *modonl*, is represented together with results available in the literature (Bennouna and White, 1984; Benamar et al., 1991). To perform model reduction, only the first mode was initially considered. When the regenerated mode is determined, the modal-oscillator equation is seen to obey the following general pattern:

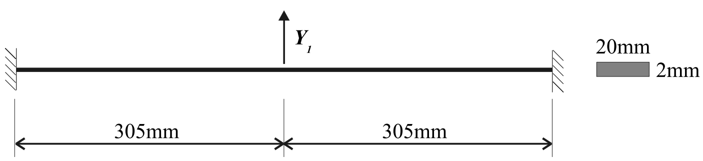

Fig. 6. Clamped–clamped beam characteristics: $E = 7.33 \times 10^{10}$ N/m^2, $\rho = 2770$ kg/m^3.

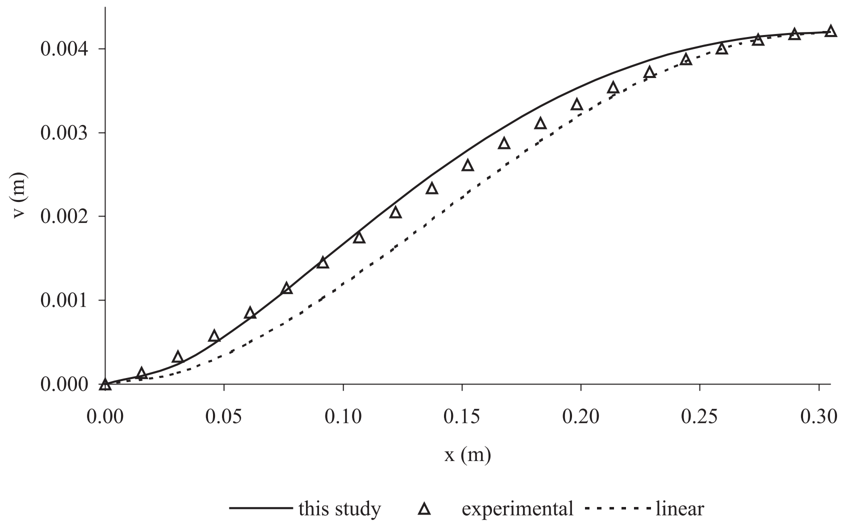

Fig. 7. First mode of the clamped–clamped beam.

$$\ddot{Y}_1 + aY_1 + cY_1Y_1Y_1 + d\dot{Y}_1\dot{Y}_1Y_1 = 0. \quad (56)$$

The frequency–amplitude relationship can be established using the multiple scales method and it is approximately given by

$$\frac{\omega}{\sqrt{a}} = 1 + \frac{3c}{8a}A^2. \quad (57)$$

To assess the quality of the model with a single modal degree of freedom, another reduced model of higher hierarchy was sought. The second symmetric non-linear mode, Fig. 8, was also used in the reduction

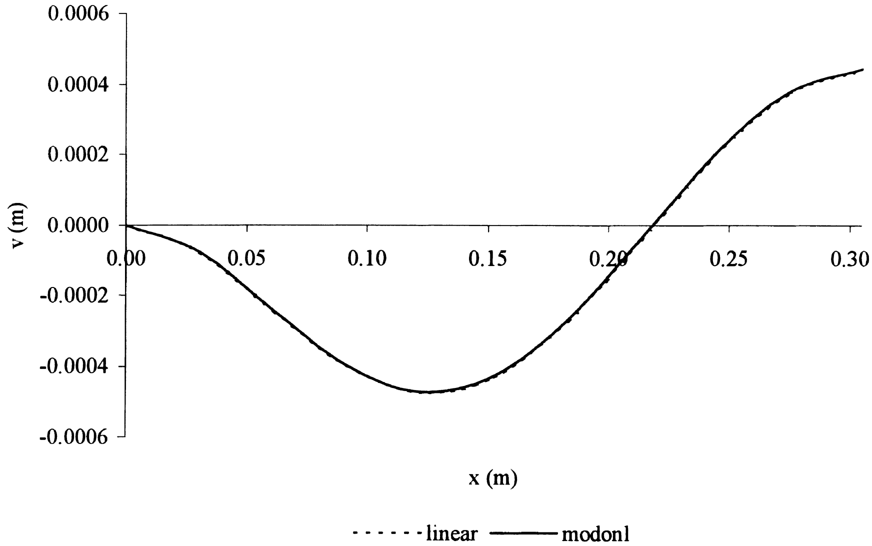

Fig. 8. Second symmetric mode of the clamped–clamped beam.

Table 3
Comparison among models for the first mode of the clamped–clamped beam

	Original model	Regenerated (1 dof)	Regenerated (2 dof)
a	31889	31889	31889
c	5.37×10^9	5.46×10^9	5.45×10^9
d	34482	26638	26639
$3c/(8a)$	63123	64179	64102
Δ (%)		1.67	1.55

technique and its effect on the first modal oscillator equation was then assessed. Table 3 shows that the improvement is not significant, as the Δ deviation with respect to the original modal oscillator equation, which was already small, is not greatly reduced further.

7. Validation tests in non-linear forced vibration

Validation tests in non-linear forced vibration are still to be devised in more depth before they can be performed. One possible strategy would be the comparison between the reduced model and the full finite-element model, as far as the response results for specific situations are concerned. At this moment, such tests have not been carried out, since a new version of the finite-element code based upon the same formulation is being constructed at the Computational Mechanics Laboratory. Outputs of largely used finite-element codes are not directly comparable to ours, as the non-linear formulations are not strictly the same. It is expected that the proper orthogonal decomposition technique, or the Karhunen–Loeve method, may be of considerable relevance in the quality assessment of reduced models, as it allows for the determination of the energy content ratio in the selected modes.

8. Conclusions

The paper represents a contribution towards the automatic characterisation of reduced models in non-linear dynamics, once full finite-element models of the structural system are available. For the time being only planar frames can be tackled, although in many respects the procedures discussed here are applicable to other structural systems. It is a pioneer research work, as the non-linear modal analysis and the reduction technique are applied to systems of distributed properties discretised by the finite-element method.

A remarkable feature of the reduction technique is that it was devised to be performed finite element by finite element, as opposed to manipulating the full model. Further, matrix operations may be carried out in the finite-element local system.

The reduction technique usually leads to coupled equations, even though the non-linear singular modes used as input are orthogonal, therefore allowing for assessing modal interaction under specific dynamic excitation conditions.

Under internal resonance conditions, non-linear singular modes are not properly defined and multi-modes should be determined to be used as input for the reduction technique.

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